

V. *On the Nature of Negative and of Imaginary Quantities.*By DAVIES GILBERT, *Esq. P.R.S.*

Read November 18, 1830.

I AM desirous of submitting to the Royal Society some observations on the nature of what are termed Negative and Imaginary Quantities, tending as I hope to clear away the obscurity that has hitherto surrounded them.

The subject has occupied my attention for many years, and however plain and simple may be the results, they have not been obtained without much patient investigation; and, in the event of their being found correct, they will add one authority more to an observation frequently made, and confirmed by extensive experience,—That paradoxes and apparent solecisms, involving themselves with facts or with deductions known to be true, may always be found near the surface, owing their existence either to ambiguities of expression or to the unperceived adoption of some extraneous additions or limitations into the compound terms used for definition, which are subsequently taken as constituent parts of their essence.

The first misapprehension appears to consist in our considering any quantity whatever as negative per se, and without reference to another opposed to it, which has previously been established as positive.

In applying our arithmetic to whatever is continuous, some neutral point or zero must be selected, as in time, in space with reference to its three dimensions, in forces, in velocities; and the opposite directions from this point will be mutually negative in respect to each other, and must be distinguished by appropriate marks or signs. But space to the left is no more essentially negative than space to the right, nor descent than ascent, nor time past than time that is to come.

I would therefore adopt for the present investigation, and to avoid previously formed association of ideas, (*a*) and (*b*) as marks or signs for prefixing

to the same quantity taken in opposite directions, rather than the usual ones of plus and minus.

In the next place, I believe that the law of the signs has never been stated according to its full generalization.

In common language, and for ordinary purposes, multiplication is considered as an abbreviated addition; but it would be a superfluous waste of time to demonstrate before this Society that multiplication is always an affair of ratios. Length and breadth multiplied together give areas, because an unit of length by an unit of breadth has previously been established as the superficial unit. Length, width, and depth give solidity for the same reason, and from the want of such a preestablished unit, arises the utter absurdity of a question, heretofore proposed in various treatises on arithmetic, for multiplying some denomination of coin by itself, and ascertaining the product.

When a multiplication of two quantities is therefore to be made, unity must be understood as the antecedent; but here an extraneous limitation insinuates itself unperceived. The common antecedent taken in usual practice is not simply an unit, but unity in the scale of (*a*). With this limitation the law of the signs is correct, namely that like signs produce (*a*), and unlike signs produce (*b*). But let unity, the common antecedent, be taken in the scale of (*b*); the law will then immediately change to like signs producing (*b*), and unlike signs producing (*a*).

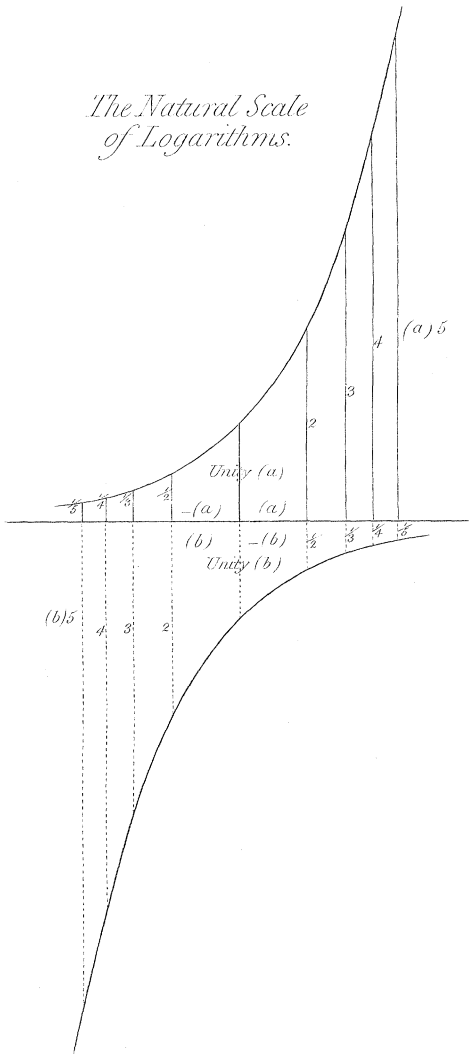
The general rule therefore must be, that like signs give the sign of the assumed universal antecedent, and unlike signs the contrary.

Admitting, therefore, that both scales are in themselves equally affirmative, and that either may be taken as negative to the other, it necessarily follows that by using the scale of (*b*), and consequently by assuming the unit of that scale as the universal antecedent, all even roots in the scale of (*a*) will become imaginary; and thus the apparent discrimination of the two scales is entirely removed: and in the same way, and by varying the signs according to the scale in which the universal antecedent is taken, all formulæ will become equally applicable to both.

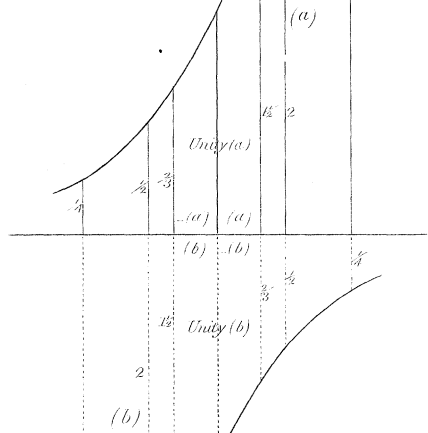
For example: (See plate III.)

The natural numbers and their logarithms will be expressed for both scales by the correlative curves in the following figures, where all ordinates taken

The Natural Scale of Logarithms.



The Tabular Scale of Logarithms.



continuously above and below the line of the apsides are reciprocal to each other; and their product equal to the square of either ordinate at the zero point, which, in the position of the curves corresponding to the natural system, is when the fluxions of x and y are equal.

It is obvious from inspecting this figure, that if m be a natural number and M its logarithm, both in the scale of (a) , and M be taken in the scale of (b) , or $-(a)$, it will become the logarithm of $\frac{1}{m}$; but M in the scale of (b) is also the logarithm of m in the same scale; and M taken as at first in the scale of (a) , or $-(b)$, is the logarithm of $\frac{1}{m}$.

If the two curves are moved on each other, so that the two ordinates measuring 2.302585 in reference to the former unit be made continuous, and that length be taken as a new unit, they will then represent the tabular logarithms in both scales.

I flatter myself with having now clearly established the principle, that all properties belonging to the scales of (a) and (b) are mutually interchangeable; and that consequently imaginary quantities will be found in the even roots of either scale, as the universal antecedent is taken in the other. And this leads to the question,

What are imaginary quantities?

I answer, Creations merely of arbitrary definition, endowed with properties at the pleasure of him that defines them, the whole dispute respecting imaginary quantities turning on the point contested from the earliest times between the hostile sects of Realists and Nominalists, descending through Plato, Aristotle and the Stoics, to the Philosophers of Alexandria, and from them to the Schoolmen, who imbittered their discussions with theological controversy and persecution.

If the conceptions of the mind, all abstractions and generalizations, were considered as substantial forms, possessing existences distinct from the intelligences contemplating them; or, as some writers have expressed themselves *autousia gaudentes*, the *μορφαι* immutable and eternal, which united to matter constitute the universe, nothing could be equally absurd with the supposition of impossible or imaginary quantities; but according to the theory now so uni-

versally prevalent as scarcely to admit of a single exception among all those who make the powers of the human mind the subject of their peculiar research; Classifications, Abstractions, Generalizations are allowed to be mere creatures of the reasoning faculty, existing nowhere but in the mind by which they are contemplated. To such unsubstantial existences any qualities may be imputed; but the only one known or useful in algebra, is the supposed even root of a real quantity taken in the scale opposite to that which has given the universal antecedent. The sign or mark indicating the extraction impossible to be performed, veils the real quantity, and renders it of no actual value until the sign is taken away by an involution, or the reverse of the supposed operation which that mark or sign represents, although by its arbitrary essence the quantity so veiled is in the mean time made applicable to all the purposes for which real quantities are used in all kinds of formulæ.

While therefore the sign of the supposed extraction of a root remains, the quantity to which it is prefixed has no more than a potential existence; but it stands ready to exist in energy whenever that sign is removed.

Consequently, without experience, it is impossible to know whether an implicit function of such an ideal quantity, will or will not be cleared by development of the symbol indicating the supposed extraction of a root, that is, whether any actual value does or does not belong to such a function.

Subject to the above conditions, namely, that the quantity veiled by the sign of a supposed extraction shall be treated in expansions and formulæ according to the laws applicable to real quantities, and that it shall exist in energy whenever an involution has reversed the supposed extraction of an even root,—

Let (A) be supposed equal to $\sqrt{-1} \sqrt{-1}$ to find (A);

Then, according to the established laws of real quantities arbitrarily extended to these that are imaginary, the log. of A = $\sqrt{-1} \times$ log. of $\sqrt{-1}$; but by a well known theorem,

$$\begin{aligned} \text{the log. of } \sqrt{-1} &= \left(\sqrt{-1} - \frac{1}{\sqrt{-1}} \right) - \frac{1}{2} \left(\sqrt{-1}^2 - \frac{1}{\sqrt{-1}^2} \right) \\ &+ \frac{1}{3} \left(\sqrt{-1}^3 - \frac{1}{\sqrt{-1}^3} \right) - \frac{1}{4} \left(\sqrt{-1}^4 - \frac{1}{\sqrt{-1}^4} \right) \&c. \end{aligned}$$

And this series $\times \sqrt{-1}$ becomes $-2 \times \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} \&c.$ each alternate term

vanishing. But $2 - \frac{2}{3} + \frac{2}{3} + \frac{2}{7} - \frac{2}{9}$ &c. = the quadrantal arc of a circle to radius unity.

Therefore the log. of $A = -$ quad. arc.

$$\text{And } A = e^{-\text{quad. arc}} = 0.2078796.$$

Consequently, $\sqrt{-1}^{\sqrt{-1}}$ is an abbreviated mark or symbol, according to the above arbitrary conditions, for the radix of the natural system of logarithms raised to a negative power, indicated by the quadrantal arc of a circle to radius unity. And in the event of $\sqrt{-1}^{\sqrt{-1}}$ ever occurring in the solution of a problem, $e^{-\text{quad. arc}}$ or $2.71828^{-1.5708}$ or 0.2078796 may be substituted for it. And this is what practically happens in regard to all expressions apparently imaginary, which are found to represent real quantities, as is well known in cases of circular arcs and logarithms. These mental abstractions have moreover extended the bounds of analysis far beyond the utmost limits it could otherwise have attained; they have bestowed harmony and simplicity of form on its most recondite investigations, and eminently has their use been important in equations, by resolving them into a number of simple factors equal to the dimensions of the equation in its highest term.

It appears from these considerations, that several ingenious mathematicians have taken an incorrect view of ideal quantities, by mistaking incidental properties for those which constitute their essence; as, for example, when they are supposed to be principles of perpendicularity, because they may in some cases indicate extension at right angles to the direction here designated by (*a*) and (*b*), but with an equal degree of propriety might the actually existing square root of a quantity be considered as the principle of obliquity; because, in certain cases, it indicates the hypotenuse of a right-angled triangle.

I would here notice an error in reasoning (as it appears to me), fallen into by all authors who have endeavoured to explain the mode of arriving at a true conclusion respecting the sines and cosines of multiple arcs; which reasoning imputes actual properties to ideal quantities, instead of deriving them all from mere arbitrary convention.

Given the sine, and consequently the cosine of an arc, to find the sine and cosine of *n* times that arc:

Let z the original arc, v the sine, and y the cosine, x the cosine of the arc $n z$, then as $\dot{z} : -\dot{y} :: 1 : \sqrt{1-y^2} \therefore \dot{z} = \frac{-\dot{y}}{\sqrt{1-y^2}}$ therefore

$$\frac{\dot{x}}{\sqrt{1-x^2}} = n \cdot \frac{\dot{y}}{\sqrt{1-y^2}}$$

No integration can however be effected of these quantities in their actual form; but changing the signs of the terms in both denominators,

$$\frac{\dot{x}}{\sqrt{x^2-1}} = n \cdot \frac{\dot{y}}{\sqrt{y^2-1}} \text{ and}$$

the nat. log. of $(x + \sqrt{x^2-1}) = n \times$ the nat. log. of $(y + \sqrt{y^2-1})$

and $x + \sqrt{x^2-1} = (y + \sqrt{y^2-1})^n = y^n + n y^{n-1} \cdot \sqrt{y^2-1}$

$$+ n \frac{n-1}{2} y^{n-2} \cdot \sqrt{y^2-1} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot y^{n-3} \cdot \sqrt[3]{y^2-1} \text{ \&c.}$$

But since y is taken as the cosine of the original arc, and x is the cosine of the multiple arc, and consequently each less than unity, it is obvious that the second term on the left of the equation, and that every even term of the expansion on the right, can exist only in the potential form of an ideal quantity; and a conclusion has thence been drawn (but as it seems to me on no solid principle whatever), that since real and imaginary quantities occur on each side of the equation, and they are of a nature completely heterogeneous one to the other, each must be respectively equal; but this mode of reasoning clearly imputes qualities to mere symbols beyond those originally imparted to them. The double equality, on my principles, depends entirely on its being assumed; as in the solution of cubic equations.

When $a + b$ have been substituted for x in the equation $x^3 - qx + r = 0$, and it becomes changed into $a^3 + b^3 + (3ab - q) \times \overline{a+b} + r = 0$, two unknown quantities exist with but one relation; another may therefore be assumed; and that which obviously reduces the expression to the most simple form is obtained by making $3ab - q = 0$.

In the same manner x and y , the cosines of two arcs, having but one relation, admit of another being assumed; any relation might be taken, but the one clearly indicated is that which makes the real terms on both sides equal:

this assumption leaves the ideal quantities in their actual state without any change, and when the sign of an imaginary operation has been removed in the usual way, they also become truly equal on each side of the equation, and from this double equality, series readily present themselves expressing the sine and cosine of the multiple are required.

On the whole, it appears to me that mystery and paradox are entirely banished from the science most powerful in eliciting truth, and where they ought least to be found,—by considering all quantity as affirmative per se, and admitting plus and minus only as correlative terms; consequently, by extending the law of the signs, so as to make the multiplication of like signs give that of the scale in which the universal antecedent has been assumed, and the multiplication of unlike signs the contrary; and, finally, by excluding all actual existence, and thereby all inherent properties, from the symbols of quantities veiled by the mark indicative of an operation incapable of being performed, but arbitrarily endowing them with the properties of real quantities in all expansions and formulæ, and with the ultimate quality of regaining their actual existence whenever the veil is removed, by an operation the inverse of that which had originally induced it.